Additional materials to the article: Weakly nonlinear analysis of thermoacoustic instabilities in annular combustors

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Appendix A. Averaging with describing functions

This appendix shows how to evaluate the expressions (3.4) for $k = 1$, with the case $k = 2$ following similarly. We substitute the definition of f_1 from (2.34a), and consider only the j-th term of the summation in (3.4):

$$\frac{c_j}{2}\frac{1}{\pi/\omega}\int_0^{2\pi/\omega} \mathcal{Q}\left[A_1 c_j \cos(\omega t+\varphi_1) + A_2 s_j \cos(\omega t+\varphi_2)\right] e^{i(\omega t+\varphi_1)} dt \tag{A 1}$$

We introduce the first change of variables

$$\begin{cases} a_j = A_1 c_j \cos\varphi_1 + A_2 s_j \cos\varphi_2 \\ b_j = A_1 c_j \sin\varphi_1 + A_2 s_j \sin\varphi_2 \end{cases} \tag{A 2}$$

and the second change of variables

$$\begin{cases} a_j = R_j \cos\psi_j \\ b_j = R_j \sin\psi_j \end{cases} \qquad \begin{cases} R_j = & \sqrt{a_j^2 + b_j^2} \\ \psi_j = & \arg\left(a_j + i b_j\right) \end{cases} \tag{A 3}$$

Notice that we can rewrite the definition of R_j by substituing the expression for a_j, b_j from (A 2), obtaining equation (3.6). We trigonometrically expand the argument of $\mathcal{Q}'$ in equation (A 1), and substitute first (A 2) and then (A 3). The expression (A 1) simplifies to

$$\frac{c_j}{2}\frac{1}{\pi/\omega}\int_0^{2\pi/\omega} \mathcal{Q}\left[R_j \cos(\omega t+\psi_j)\right] e^{i(\omega t+\varphi_1)} dt \tag{A 4}$$

We first change the time variable to $t \to t - \psi_j/\omega$ and then slide the interval of definition of the integrand because it is periodic. We obtain

$$\frac{c_j}{2}\frac{1}{\pi/\omega}\int_0^{2\pi/\omega} \mathcal{Q}\left[R_j \cos(\omega t)\right] e^{i(\omega t+\varphi_1-\psi_j)} dt \tag{A 5}$$

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We then expand the complex exponential in the integrand, take the constants out of the integral, and divide and multiply by R_j:

$$R_j e^{-i\psi_j} \frac{c_j}{2} e^{i\varphi_1} \left[\frac{1}{R_j} \frac{1}{\pi/\omega} \int_0^{2\pi/\omega} \mathcal{Q}\left[R_j \cos(\omega t)\right] e^{i\omega t} dt \right] \tag{A 6}$$

We now observe that the term in the outer square brackets is the describing function of $\mathcal{Q}$ defined in (2.4). From (A 3) we have that $R_j e^{-i\psi_j} = a_j - ib_j$. The expression simplifies to

$$(a_j - ib_j)\frac{c_j}{2} e^{i\varphi_1} G(R_j, \omega) e^{i\phi(R_j,\omega)} \tag{A 7}$$

where we have rewritten the describing function $Q(R_j, \omega)$ in terms of gain G and phase response ϕ as presented in equation (2.5). This is the contribution of the j-th burner. The final expression of (3.4) is

$$\langle f_k \cos(\omega t + \varphi_k)\rangle + i\langle f_k \sin(\omega t + \varphi_k)\rangle = \sum_{j=1}^{N_b} (a_j - ib_j)\frac{c_j}{2} G(R_j, \omega) e^{i(\phi(R_j,\omega)+\varphi_k)} \tag{A 8}$$

The two averaged terms are the real and imaginary parts of (A 8). By substituting a_j, b_j from (A 2) we obtain:

$$\langle f_1 \cos(\omega t + \varphi_1)\rangle = +\frac{1}{2}\sum_{j=1}^{N_b} G(R_j, \omega)\left[A_1 c_j^2 \cos\phi(R_j, \omega) + A_2 c_j s_j \cos(\phi(R_j, \omega) + \varphi)\right] \tag{A 9a}$$

$$\langle f_1 \sin(\omega t + \varphi_1)\rangle = +\frac{1}{2}\sum_{j=1}^{N_b} G(R_j, \omega)\left[A_1 c_j^2 \sin\phi(R_j, \omega) + A_2 c_j s_j \sin(\phi(R_j, \omega) + \varphi)\right] \tag{A 9b}$$

Similary for $j = 2$ we obtain

$$\langle f_2 \cos(\omega t + \varphi_1)\rangle = +\frac{1}{2}\sum_{j=1}^{N_b} G(R_j, \omega)\left[A_2 s_j^2 \cos\phi(R_j, \omega) + A_1 c_j s_j \cos(\phi(R_j, \omega) - \varphi)\right] \tag{A 9c}$$

$$\langle f_2 \sin(\omega t + \varphi_1)\rangle = +\frac{1}{2}\sum_{j=1}^{N_b} G(R_j, \omega)\left[A_2 s_j^2 \sin\phi(R_j, \omega) + A_1 c_j s_j \sin(\phi(R_j, \omega) - \varphi)\right] \tag{A 9d}$$

Finally, by substituting (A 9) in (3.3), we obtain the slow flow equations (3.5).

Appendix B. Sufficient condition for the existence of fixed points

This appendix proves the implication (3.15). We first introduce some simple mathematical identities, and then provide the proof in §B.2.

B.1. *Mathematical identities*

The following properties hold for any function $f:\mathbb{R}\rightarrow\mathbb{R}$:

$$\sum_{j=1}^{N_b} c_j^2 f\left(R_j(A_2,A_1,-\varphi)\right) = \sum_{j=1}^{N_b} s_j^2 f\left(R_j(A_1,A_2,\varphi)\right) \tag{B 1a}$$

$$\sum_{j=1}^{N_b} \cos(2\theta_j) f(R_j) = 0 \qquad \forall\,\varphi\,,\, A_1 = A_2 \tag{B 1b}$$

$$\sum_{j=1}^{N_b} \sin(2\theta_j) f(R_j) = 0 \qquad \varphi = \pm\frac{\pi}{2}\,,\, A_1 = A_2 \tag{B 1c}$$

$$\sum_{j=1}^{N_b} c_j^2 = \sum_{j=1}^{N_b} s_j^2 = \frac{N_b}{2} \qquad , \qquad \sum_{j=1}^{N_b} c_j s_j = 0 \tag{B 1d}$$

where c_j and s_j are defined in (2.17).

B.2. *Proof of* (3.15)

The proof proceeds following these steps. In §B.2.1 we first prove that $f_\varphi(A,A,k\pi/2)$ is zero for all integer values of k. We then prove in §B.2.2 that $f_{A_1}(A,A,k\pi/2) - f_{A_2}(A,A,k\pi/2)$ is zero. It follows that if also $f_{A_1}(A,A,k\pi/2) + f_{A_2}(A,A,k\pi/2)$ is zero, as is the hypothesis of the implication, then the individual terms, $f_{A_1}(A,A,k\pi/2)$ and $f_{A_2}(A,A,k\pi/2)$ also have to be zero. Then, all the three functions $f_\varphi, f_{A_1}, f_{A_2}$ are zero, and the point $(A,A,k\pi/2)$ is therefore a fixed point.

B.2.1. *First part: $f_\varphi = 0$*

The function $f_\varphi(A,A,\varphi)$ is odd with respect to φ. It follows that $\varphi=0$ is a zero of the function:

$$f_\varphi(A,A,2k\pi) = 0\ \forall k \in \mathbb{Z} \tag{B 2}$$

By direct substitution, we can show that (B 2) holds also at $\varphi=\pm\pi/2$ and $\varphi=\pi$:

$$f_\varphi(A,A,k\pi/2) = 0\ \forall k \in \mathbb{Z} \tag{B 3}$$

We prove (B 3):

- for $\varphi=\pi/2$ by observing that R_j is independent of j. The expression becomes

$$f_\varphi(A,A,\pm\pi/2) = \frac{1}{2}\sum_{j=1}^{N_b}\left[\left(s_j^2 - c_j^2\right)\sin\phi\left(A,\omega\right) - 2c_j s_j\cos\phi\right] G\left(A,\omega\right) \tag{B 4}$$

This equation (B 4) can be split into two summations of $\cos 2\theta_j$ and $\sin 2\theta_j$, which are zero when summed over $[0\,,\,2\pi]$ as can be deduced from (B 1d):

- for $\varphi=\pi$ in (B 3) we obtain

$$\begin{aligned} f_\varphi(A,A,\pi) &= \frac{1}{2}\sum_{j=1}^{N_b}\left(s_j^2 - c_j^2\right)\sin\phi(R_j,\omega) G\left(R_j,\omega\right) \\ &= -\frac{1}{2}\sum_{j=1}^{N_b}\cos(2\theta_j)\sin\phi\left(R_j,\omega\right) G\left(R_j,\omega\right) \end{aligned} \tag{B 5}$$

This summation vanishes by applying the property (B 1b).

B.2.2. *Second part:* $f_{A_1} - f_{A_2} = 0$

We now prove that

$$f_{A_1}(A, A, k\pi/2) - f_{A_2}(A, A, k\pi/2) = 0 \;\forall k \in \mathbb{Z} \tag{B 6}$$

1) for $\varphi = 0$ by direct substitution and by exploiting (B 1a); *2)* for $\varphi = \pi$, by direct substitution and exploiting (B 1a) for $A_1 = A_2$ in the resulting equation; for $\varphi = \pi/2$ by also applying (B 1c) twice. We also observe that

$$\begin{cases} f_{A_1}(A, A, k\pi/2) = & 0 \\ f_{A_2}(A, A, k\pi/2) = & 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} f_{A_1}(A, A, k\pi/2) - f_{A_2}(A, A, k\pi/2) = & 0 \\ f_{A_1}(A, A, k\pi/2) + f_{A_2}(A, A, k\pi/2) = & 0 \end{cases} \tag{B 7}$$

This, together with (B 6) and (B 3) implies that

$$f_{A_1}(A, A, k\pi/2) + f_{A_2}(A, A, k\pi/2) = 0 \quad \Rightarrow \quad \boldsymbol{f}(A, A, k\pi/2) = 0 \tag{B 8}$$

Appendix C. Conditions for the stability of fixed points

One can numerically calculate the amplitudes A^{sp} of spinning solutions from equation (3.16) and the amplitudes A^{st} of standing solutions from equation (3.19). We can then discuss the stability of these solutions by evaluating the eigenvalues of the Jacobian of the system (3.5). If all eigenvalues are negative the point is an attractor, i.e. a stable solution. Since the eigenvalues are invariant with respect to a change of variables, we consider the new set of variables

$$\begin{cases} 2C = A_1 + A_2 \\ 2D = A_1 - A_2 \end{cases} \qquad (\mathrm{C}\,1a) \qquad\qquad \begin{cases} A_1 = C + D \\ A_2 = C - D \end{cases} \qquad (\mathrm{C}\,1b)$$

This transformation maps the point $(A_1, A_2, \varphi) = (A, A, \varphi)$ to the point $(C, D, \varphi) = (A, 0, \varphi)$, and the reason we apply the transformation will be apparent later. By evaluating the time derivative of (C 1a) and substituting first (3.5) and then (C 1b), we obtain the slow flow in terms of the new variables:

$$C' = -\frac{\alpha}{2}C + \frac{1}{4}\sum_{j=1}^{N_b}\Big[\big((c_j^2 - s_j^2)D + C\big)\cos\phi\,(R_j,\omega) + c_j s_j (C - D)\cos\left(\phi\,(R_j,\omega) + \varphi\right) + c_j s_j (C + D)\cos(\phi\,(R_j,\omega) - \varphi)\Big]G\,(R_j,\omega) \tag{C 2a}$$

$$D' = -\frac{\alpha}{2}D + \frac{1}{4}\sum_{j=1}^{N_b}\Big[\big((c_j^2 - s_j^2)C + D\big)\cos\phi\,(R_j,\omega) + c_j s_j (C - D)\cos(\phi\,(R_j,\omega) + \varphi) - c_j s_j (C + D)\cos(\phi\,(R_j,\omega) - \varphi)\Big]G\,(R_j,\omega) \tag{C 2b}$$

$$\varphi' = \frac{1}{2}\sum_{j=1}^{N_b}\left[\left(s_j^2 - c_j^2\right)\sin\phi\,(R_j,\omega) - c_j s_j\left(\frac{C - D}{C + D}\sin(\phi\,(R_j,\omega) + \varphi) - \frac{C + D}{C - D}\sin(\phi\,(R_j,\omega) - \varphi)\right)\right]G\,(R_j,\omega) \tag{C 2c}$$

We can rewrite the system (C 2) in compact form, and study the gradients of f_C, f_D, f_φ at the position of the fixed points found in §3.3, to obtain the Jacobian matrix $\boldsymbol{J}$:

$$\begin{cases} C' & = f_C(C, D, \varphi) \\ D' & = f_D(C, D, \varphi) \\ \varphi' & = f_\varphi(C, D, \varphi) \end{cases} \qquad \boldsymbol{J} = \begin{bmatrix} \frac{\partial f_C}{\partial C} & \frac{\partial f_C}{\partial D} & \frac{\partial f_C}{\partial \varphi} \\ \frac{\partial f_D}{\partial C} & \frac{\partial f_D}{\partial D} & \frac{\partial f_D}{\partial \varphi} \\ \frac{\partial f_\varphi}{\partial C} & \frac{\partial f_\varphi}{\partial D} & \frac{\partial f_\varphi}{\partial \varphi} \end{bmatrix} \tag{C 3}$$

The eigenvalues of $\boldsymbol{J}$ will allow the stability of the fixed points to be assessed. In order to evaluate this, we need to study the dependence of R_j as a function of C, D, φ. Applying the change of variables (C 1*b*) to the definition (3.6) of R_j, and then setting $D = 0$, we obtain

$$R_j = C\sqrt{1 + 2c_j s_j \cos\varphi} \tag{C 4}$$

The derivatives of R_j with respect to C, D, φ at a point $(C, D = 0, \varphi)$ are

$$\begin{cases} \frac{\partial R_j}{\partial C} & = \sqrt{1 + 2c_j s_j \cos\varphi} \\ \frac{\partial R_j}{\partial D} & = \frac{c_j^2 - s_j^2}{\sqrt{1+2c_j s_j \cos\varphi}} \\ \frac{\partial R_j}{\partial \varphi} & = -\frac{c_j s_j \sin\varphi}{\sqrt{1+2c_j s_j \cos\varphi}} C \end{cases} \tag{C 5a}$$

For a spinning wave, i.e. setting $(C, D, \varphi) = (A^{sp}, 0, \pi/2)$ and a standing wave, i.e. setting $(C, D, \varphi) = (A^{st}, 0, 0)$, equations (C 4) and (C 5) become

$$\text{spinn. wave:} \begin{cases} R_j & = A^{sp} \\ \frac{\partial R_j}{\partial C} & = 1 \\ \frac{\partial R_j}{\partial D} & = c_j^2 - s_j^2 \\ \frac{\partial R_j}{\partial \varphi} & = -c_j s_j A^{sp} \end{cases} \quad \text{stand. wave:} \begin{cases} R_j & = A^{st}\sqrt{1 + 2c_j s_j} \\ \frac{\partial R_j}{\partial C} & = \sqrt{1 + 2c_j s_j} \\ \frac{\partial R_j}{\partial D} & = \frac{c_j^2 - s_j^2}{\sqrt{1+2c_j s_j}} \\ \frac{\partial R_j}{\partial \varphi} & = 0 \end{cases} \tag{C 6}$$

From now onwards, we will use a subscript *sp* to denote that a quantity is evaluated at the fixed point of a spinning wave, and a subscript *st* to denote that a quantity is evaluated at the fixed point of a standing wave.

In evaluating the terms of (C 3), one first analytically evaluates the gradients, and then substitute $(C, D, \varphi) = (A^{sp}, 0, \pi/2)$ for spinning solutions and $(C, D, \varphi) = (A^{st}, 0, 0)$ for standing solutions, and then equation (C 6). For both standing and spinning waves the Jacobian is a block diagonal matrix:

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial f_C}{\partial C} & 0 & 0 \\ 0 & \frac{\partial f_D}{\partial D} & \frac{\partial f_D}{\partial \varphi} \\ 0 & \frac{\partial f_\varphi}{\partial D} & \frac{\partial f_\varphi}{\partial \varphi} \end{bmatrix} \tag{C 7}$$

This was expected from the symmetries of the equations and is the reason why we applied the change of variables (C 1*b*). One eigenvalue is trivially $\lambda_1 = \frac{\partial f_C}{\partial C}$, and can be interpreted in terms of the Rayleigh criterion at limit-cycles, as discussed in the main manuscript. The other 2 eigenvalues are the solutions of the characteristic polynomial

$$\lambda^2 - \left(\frac{\partial f_D}{\partial D} + \frac{\partial f_\varphi}{\partial \varphi}\right)\lambda + \left(\frac{\partial f_D}{\partial D}\frac{\partial f_\varphi}{\partial \varphi} - \frac{\partial f_D}{\partial \varphi}\frac{\partial f_\varphi}{\partial D}\right) = 0 \tag{C 8}$$

Applying the Routh-Hurwitz criterion, all the real parts of the three eigenvalues are not negative, i.e. the fixed point is stable or neutrally stable, if and only if all the coefficients of the second order polynomial (C 8) are not negative (Hurwitz 1964). This leads to the following necessary and sufficient conditions for stability:

$$\begin{cases} \frac{\partial f_C}{\partial C} & \leq 0 \\ \frac{\partial f_D}{\partial D} + \frac{\partial f_\varphi}{\partial \varphi} & \leq 0 \\ \frac{\partial f_D}{\partial D}\frac{\partial f_\varphi}{\partial \varphi} - \frac{\partial f_D}{\partial \varphi}\frac{\partial f_\varphi}{\partial D} & \geq 0 \end{cases} \tag{C 9}$$

C.1. *Stability of spinning solutions*

For a spinning solution $(C, D, \varphi) = (A^{sp}, 0, \pi/2)$, the 5 components of the Jacobian (C 7) are:

$$\frac{\partial f_C}{\partial C}_{sp} = \frac{A^{sp}}{2} F^{sp'}\left(A^{sp}\right) \tag{C 10a}$$

$$\frac{\partial f_D}{\partial D}_{sp} = \frac{A^{sp}}{4} F^{sp'}\left(A^{sp}\right) \tag{C 10b}$$

$$\frac{\partial f_D}{\partial \varphi}_{sp} = \frac{{A^{sp}}^2}{16} N_b \mathrm{Im}\left[Q'\left(A^{sp}, \omega\right)\right] \tag{C 10c}$$

$$\frac{\partial f_\varphi}{\partial D}_{sp} = -\frac{N_b}{4} \mathrm{Im}[Q'\left(A^{sp}, \omega\right)] \tag{C 10d}$$

$$\frac{\partial f_\varphi}{\partial \varphi}_{sp} = \frac{A^{sp}}{4} F^{sp'}\left(A^{sp}\right) \tag{C 10e}$$

where the prime expresses a derivative with respect to the amplitude A. The stability conditions (C 9) for a spinning mode are:

$$F^{sp'}\left(A^{sp}\right) < 0 \tag{C 11a}$$

$$\frac{A^{sp}}{4} F^{sp'}\left(A^{sp}\right) + \frac{A^{sp}}{4} F^{sp'}\left(A^{sp}\right) < 0 \tag{C 11b}$$

$$\left(\frac{A^{sp}}{4} F^{sp'}\left(A^{sp}\right)\right)^2 - \frac{{A^{sp}}^2}{16} N_b \mathrm{Im}\left[Q'\left(A^{sp}, \omega\right)\right] \left(-\frac{N_b}{4} \mathrm{Im}[Q'\left(A^{sp}, \omega\right)]\right) > 0 \tag{C 11c}$$

Trivially the second inequality is equivalent to the first one. We substitute in the first and third inequality the definition of F^{sp} from (3.17), and we obtain:

$$\mathrm{Re}\left[Q'\left(A^{sp}, \omega\right)\right] < 0 \tag{C 12}$$

$$\mathrm{Re}\left[Q'\left(A^{sp}, \omega\right)\right]^2 + \mathrm{Im}\left[Q'\left(A^{sp}, \omega\right)\right]^2 > 0 \tag{C 13}$$

with equation (C 13) always satisfied if $\mathrm{Re}\left[Q'\left(A^{sp}, \omega\right)\right] < 0$. It follows that a spinning wave with amplitude A^{sp} is stable if and only if (C 12) holds, which is the condition (3.22) reported in the manuscript.

C.2. *Stability of standing solutions*

For a standing solution $(C, D, \varphi) = (A, 0, \varphi)$, the 5 components of the Jacobian (C 7) are:

$$\frac{\partial f_C}{\partial C}_{st} = \frac{A^{st}}{2} F^{st'}\left(A^{st}\right) \tag{C 14a}$$

$$\frac{\partial f_D}{\partial D}_{st} = -\sum_{j=1}^{N_b} c_j s_j \mathrm{Re}\left[Q\left(A^{st}\sqrt{1+2c_j s_j}, \omega\right)\right] + \dots$$

$$\dots \quad A^{st}\frac{1}{4}\sum_{j=1}^{N_b} \frac{(c_j^2 - s_j^2)^2}{\sqrt{1+2c_j s_j}} \mathrm{Re}\left[Q'\left(A^{st}\sqrt{1+2c_j s_j}, \omega\right)\right] \tag{C 14b}$$

$$\frac{\partial f_D}{\partial \varphi}_{st} = -\frac{A_{st}}{2}\sum_{j=0}^{N_b} c_j s_j \mathrm{Im}\left[Q\left(A^{st}\sqrt{1+2c_j s_j}, \omega\right)\right] \tag{C 14c}$$

$$\frac{\partial f_\varphi}{\partial D}_{st} = \frac{2}{A^{st}}\sum_{j=1}^{N_b} c_j s_j \mathrm{Im}\left[Q\left(A^{st}\sqrt{1+2c_j s_j}, \omega\right)\right] - \dots$$

$$\dots \quad \frac{1}{2}\sum_{j=1}^{N_b} \frac{(c_j^2 - s_j^2)^2}{\sqrt{1+2c_j s_j}} \mathrm{Im}\left[Q'\left(A^{st}\sqrt{1+2c_j s_j}, \omega\right)\right] \tag{C 14d}$$

$$\frac{\partial f_\varphi}{\partial \varphi}_{st} = -\sum_{j=1}^{N_b} c_j s_j \mathrm{Re}\left[Q\left(A^{st}\sqrt{1+2c_j s_j}, \omega\right)\right] \tag{C 14e}$$

We substitute the expressions (C 14) in the three conditions (C 9) and obtain the inequalities (3.23) presented in the paper. In the rest of this section we prove the asymptotic properties of the inequalities (3.23) discussed in the main body of the paper. For a large number of burners N_b, we have that the sums in (3.23) can be approximated as an integral.

C.2.1. *Orientation condition*

We then discuss the second condition (3.23*b*). We observe that

$$0 = \int_0^{2\pi} \frac{\partial}{\partial \theta}\left[-\frac{\cos(2\theta)}{4}\mathrm{Re}\left[Q(A\sqrt{1+\sin(2\theta)}, \omega)\right]\right] d\theta$$

$$= \int_0^{2\pi} \frac{\sin(2\theta)}{2}\mathrm{Re}\left[Q(A^{st}\sqrt{1+\sin(2\theta)}, \omega)\right] d\theta - \dots$$

$$\dots \quad \int_0^{2\pi} A^{st}\frac{1}{4}\frac{\cos(2\theta)^2}{\sqrt{1+\sin(2\theta)}}\mathrm{Re}\left[Q'(A^{st}\sqrt{1+\sin(2\theta)}, \omega)\right] d\theta \tag{C 15}$$

The last expression of the identity (C 15) is the first term in square brackets in the first addend of (3.23*b*), which is then zero. Then also the second term in square brackets in the second addend of (3.23*b*) is zero. It follows that the whole LHS of (3.23*b*) is zero.

C.2.2. *Standing pattern condition*

We first discuss the third condition (3.23c), which becomes

$$\int_0^{2\pi} \frac{\sin(2\theta)}{2} \mathrm{Re}\left[Q(A^{st}\sqrt{1+\sin(2\theta)}, \omega)\right] d\theta - \dots$$
$$\dots \quad \int_0^{2\pi} A^{st}\frac{1}{8}\frac{\cos(2\theta)^2}{\sqrt{1+\sin(2\theta)}} \mathrm{Re}\left[Q'(A^{st}\sqrt{1+\sin(2\theta)}, \omega)\right] d\theta > 0 \tag{C 16}$$

We rewrite the second integral as

$$-\frac{1}{8}\int_0^{2\pi} \cos(2\theta)\left[\frac{A^{st}\cos(2\theta)}{\sqrt{1+\sin(2\theta)}} \mathrm{Re}\left[Q'(A^{st}\sqrt{1+\sin(2\theta)}, \omega)\right]\right] d\theta \tag{C 17}$$

where the term between square brackets is the derivative with respect to θ of the function $\mathrm{Re}\left[Q(A^{st}\sqrt{1+\sin(2\theta)}, \omega)\right]$. We integrate by parts, and obtain

$$-\frac{1}{8}\left[\left\{\cos(2\theta)\mathrm{Re}\left[Q(A^{st}\sqrt{1+\sin(2\theta)}, \omega)\right]\right\}_0^{2\pi} + \dots\right.$$
$$\left.\dots \quad 2\int_0^{2\pi} \sin(2\theta)\mathrm{Re}\left[Q(A^{st}\sqrt{1+\sin(2\theta)}, \omega)\right] d\theta\right] \tag{C 18}$$

where the first term within the square brackets evaluates to zero. We substitute this expression in (C 16) and obtain the condition (3.24).

Appendix D. Scaling of the flame response

We carry out the scaling of the gain of the flame response presented in Figure 4.b so that it is representative of existing annular combustors. We choose quite a small† growth-rate $\overline{\sigma}_r \equiv \sigma/\omega_{lin} = 0.01$ because we are studying a thermoacoustic system exhibiting triggering, so that the flame gain is weak at small amplitudes. We then exploit the fact that $\omega_0 \approx \omega_{lin}$ and fix $\overline{\alpha}_r \equiv \alpha/\omega_1 = 0.08$ similarly to Noiray *et al.* (2011), where $\alpha/\omega_0 = 0.08$. Then we can use the relation (2.32) for a given number of burners N_b and calculate a reasonable value for $\overline{\beta} \equiv \beta/\omega_{lin}$:

$$\frac{N_b\overline{\beta}\cos(\phi)}{2} = (2\overline{\sigma}_r + \overline{\alpha}_r) = 0.10 \tag{D 1}$$

where we set $\phi = \pi/5$ in the paper. $\overline{\beta}$ is the gain of the flame response at $A = 0$. This value leads to the vertical scaling of Figure 4.b for $N_b = 6$. Equation (D 1) fixes the product $\overline{\beta}N_b$ to a constant when combustors with a different number of flames are considered.

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† In Bothien *et al.* (2015) the value σ/ω_{lin} is typically of 0.05, see their figure 6.